Journal of Sound and Vibration (2003) **259**(5), 1243–1252
doi:10.1006/jsvi.2002.5288, available online at http://www.idealibrary.com on IDEAL®

APPLICATION OF SUPER ELEMENTS TO FREE VIBRATION ANALYSIS OF LAMINATED STIFFENED PLATES

M. T. AHMADIAN AND M. S. ZANGENEH

School of Mechanical Engineering, Sharif University of Technology, P.O. Box 11365-9567, *Azadi Avenue, Tehran, Iran. E-mail: ahmadian@sharif.edu*

(*Received* 5 *April* 2002, *and in final form* 24 *May* 2002)

1. INTRODUCTION

Dynamic response prediction of stiffened plates is of considerable interest for engineers. Because of their superior performance under dynamic loading, stiffened plates are extensively used in a wide range of industrial applications such as aeronautical, naval and commercial vehicle construction. Stiffened plates can be analyzed by considering equivalent orthotropic systems, but this method is limited to situations of identical, equally and closely spaced stiffeners. Another approach consists of treating separately the plates and stiffeners.

The technology evolution of structural materials has been governed by the search for a material having a higher strength to weight ratio and lower cost of fabrication. Laminated plates are fabricated by stacking up plies or lamina of unidirectional fibrous composites with each lamina oriented in the designated direction to achieve the required stiffness and strength.

Koko and Olson [1] applied a numerical technique for vibration analysis of isotropic stiffened plates. They used a super element, which is macro-element having analytical as well as the usual finite element shape functions. Koko and Olson [1–3], Jiang and Olson [4, 5], Vaziri *et al.* [6] have used this element for the analysis of plates and shells.

Of all numerical approaches, the finite element method is the most widely used method, and it has been applied with much success for the vibration analysis of plates and stiffened plates. [7–12].

The most commonly used procedure for analyzing stiffened plates using the finite element method requires that:

(1) The compatibility along the plate and stiffener boundary should be maintained by using identical shape functions while formulating the plate and beam elements.

(2) The eccentric stiffener behavior should be taken into account through a co-ordinate transformation.

(3) Stiffness and mass matrices of a beam element should be added to those of the plate elements to obtain the stiffness and mass matrices using the co-ordinate transformation.

In this paper, the dynamic analysis of stiffened plates using a super element is performed using the above procedure.

The present analysis is based on the combination of plate super elements and beam super elements based on the derivation of Koko and Olson [1]. Plate and beam super elements are presented in Figure 1. The plate and beam elements have 55 and 18 degrees of

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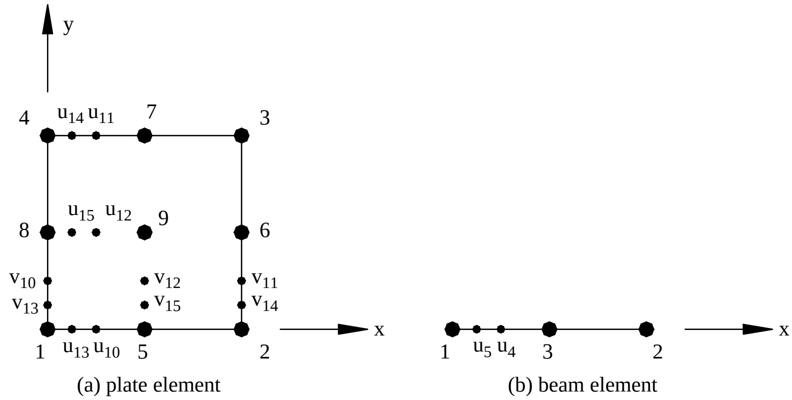

Figure 1. Super-plate element and super-beam element.

freedom respectively. The in-plane and bending shape functions of plate super element are constructed by Lagrange polynomials and Hermitian polynomials respectively. The displacement field and shape functions of these elements can be found in the literature [1,2].

2. FREE VIBRATION ANALYSIS OF LAMINATED STIFFENED PLATES

The strain energy of a rectangular symmetric cross-ply plate (with length a and width b) can be expressed as follows:

$$U_p = \frac{1}{2}\int_0^b\int_0^a [A_{11}u_{,x}^2 + 2A_{12}u_{,x}v_{,y} + A_{22}v_{,y}^2 + A_{66}(u_{,y}+v_{,x})^2 + D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2]\,\mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

where

$$A_{ij} = A_{ji} = \sum_{k=1}^{N} \bar{Q}_{\bar{ij}}^{(k)}(z_k - z_{k-1}) \quad \text{extensional stiffness of laminate,}$$

$$D_{ij} = D_{ji} = \frac{1}{3}\sum_{k=1}^{N} \bar{Q}_{\bar{ij}}^{(k)}\left(z_k^3 - z_{k-1}^3\right) \quad \text{bending stiffness of laminate,}$$

(u,v,w) are displacement field of plate and $(\,)_{,x}$ is partial derivative respect to x and $\bar{Q}_{ij}^{(k)}$ are the kth transformed reduced stiffness, N is the number of layers and z_k is the distance from kth layer to the mid-plane of the plate.

The strain energy of the beam can be obtained by summation of torsion, bending and axial strain of beam. For a beam in the x direction with 0° fiber orientation, the strain energy can be written as

$$U_b = \frac{1}{2}G_{12}J\int_0^a \theta_{,x}^2\,\mathrm{d}x + \frac{1}{2}E_1 I_{yy}\int_0^a w_{,xx}^2\,\mathrm{d}x + \frac{1}{2}E_1 A\int_0^a u_{,x}^2\,\mathrm{d}x, \qquad (2)$$

where G_{12} is the shear modulus of elasticity, E_1 the modulus of elasticity, I_{yy}, I_{zz} the area moment of inertia about y–y and z–z co-ordinate, A the cross-section area of beam, a beam length, and θ the rotation of beam.

The kinetic energy of the plate can be expressed as

$$T_p = \frac{\rho h}{2}\int_0^b\int_0^a [\dot{u}^2 + \dot{v}^2 + \dot{w}^2]\,\mathrm{d}x\,\mathrm{d}y. \tag{3}$$

where ρ is the material density and h is the plate thickness.

The kinetic energy of the beam is given by

$$T_b = \frac{\rho}{2}\int_0^b [A(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) + J_c\dot{\theta}^2]\,\mathrm{d}x. \tag{4}$$

where J_c represent the polar moment of inertia about the beam centroidal axis.

The total strain energy and kinetic energy of stiffened plate can be combined as follows:

$$\begin{aligned} U &= U_p + U_b, \\ T &= T_p + T_b. \end{aligned} \tag{5}$$

Stiffness and mass matrices can be obtained from the following formulas:

$$\begin{aligned} U &= \tfrac{1}{2}\{\mathbf{q}\}^{\mathrm{T}}[\mathbf{k}]\{\mathbf{q}\}, \\ T &= \tfrac{1}{2}\{\dot{\boldsymbol{q}}\}^{\mathrm{T}}[\mathbf{m}]\{\dot{\boldsymbol{q}}\}. \end{aligned} \tag{6}$$

where $\{\mathbf{q}\}$ and $\{\dot{\mathbf{q}}\}$ are the generalized co-ordinate and velocity respectively.

Example problems are presented in the next section.

3. NUMERICAL EXAMPLES

Different problems of isotropic and laminated stiffened plates are solved by the proposed approach in this section. The first example deals with an isotropic stiffened plate and the other examples deals with laminated plates combined with composite stiffeners. (10×10) and (2×2) meshes are used in the ANSYS and super-element analysis respectively.

3.1. CLAMPED ISOTROPIC STIFFENED PLATE

A clamped isotropic stiffened plate has been analyzed by the present super finite element method. This example, which has also been solved previously by Koko and Olson [1], is considered to validate the computer program and the methodology.

The stiffened plate is shown in Figure 2.

The material property of the plate and stiffener are as follows:

$$E = 68{\cdot}9\,\mathrm{GPa}, \quad \rho = 2670\,\mathrm{kg/m^3}, \quad v = 0.3.$$

The plate and stiffener dimensions are

$$\text{Plate}: \; a = 0{\cdot}203\,\mathrm{m}, \quad b = 0.203\,\mathrm{m}, \quad t = 0.00137\,\mathrm{m}.$$

$$\text{Stiffener}: \; h_s = 0{\cdot}01133\,m, \quad b_s = 0.00635\,\mathrm{m}.$$

A full model of the problems using four plate elements and two beam elements are considered in this study. This model has 58 net degrees of freedom (d.o.f.) and is used to enable direct comparison with a one-quarter model. The results of analysis are presented and compared in Table 1.

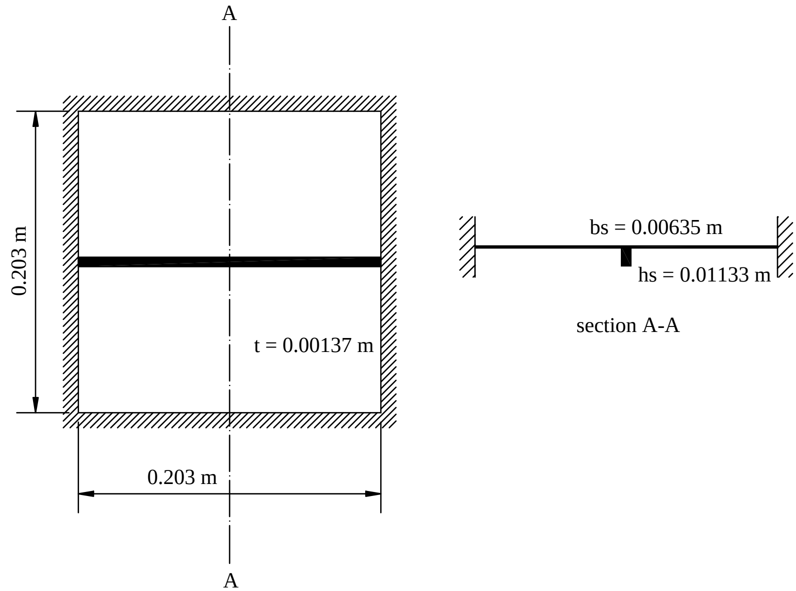

Figure 2. Stiffened isotropic plate.

TABLE 1

Comparison of natural frequencies of full model and one-quarter model of clamped isotropic stiffened plate (Hz)

Mode	Method	
	Super-full plate (Present)	Super-1/4 model sym. –sym.
1	734·8	
2	783·5	783·5
3	1019·6	
4	1035·4	
5	1485·2	
6	1491·5	1491·5
d.o.f.	58	16

Comparison of the runtime of the present model for the first six natural frequencies with a one-quarter model indicates that the application of the full model reduces the runtime approximately by one-half. This is due to the fact that in the one-quarter model, every model results in one or two natural frequencies and a complete analysis needs four one-quarter model runs. However, in the large-scale structures with numerous d.o.f., using symmetry conditions of the structures may be effective.

Comparison of full plate modelling, experimental, finite element method and ANSYS as a conventional finite element method are presented in Table 2.

As shown in Table 2, the frequencies obtained from the super element method are in agreement with the experimental and numerical results.

The effect of stiffener and eccentricity of stiffened plate are shown in Table 3.

TABLE 2

Natural frequencies of clamped isotropic stiffened plate (Hz)

Mode	Method				
	Super-element [1] one-quarter	FEM [1] one-quarter	Experimental [1]	ANSYS full model	Super-full model
1	736·8	718·1	689	685·7	734·8
2	769·4	751·4	725	782·0	783·5
3	1019·6	997·4	961	981·7	1019·6
4	1032·3	1007·1	986	1037·0	1035·4
5	1483·7	1419·8	1376	1433·5	1485·2
6	1488·3	1424·3	1413	1470·8	1491·5

TABLE 3

Effect of stiffener and eccentricity on natural frequencies of clamped isotropic stiffened plate (Hz)

Mode	Method		
	Super-full model (present)	Super (without eccentricity)	Super (without stiffener)
1	734·8	702·8	292·9
2	783·5	734·8	597·7
3	1019·6	1019·6	597·7
4	1035·4	1023·2	882·2
5	1485·2	1249·4	1100·4
6	1491·5	1485·2	1104·1

It is apparent from Table 3 that the eccentricity of the beam element sometimes plays an important role in the natural frequencies of the stiffened plate, depending on the mode number.

3.2. CLAMPED LAMINATED STIFFENED PLATE WITH STIFFENER IN THE *X* DIRECTION

The structure is made up of a square plate reinforced by a centre stiffener in the x direction. This plate, constructed from a ten plies, is clamped at the edges with a $[0°/90°/0°/90°/0°]_s$ lay-up of T300/934 CFRP material. The laminated stiffened plate is shown in Figure 3. The orthotropic material properties of the plate constituent layers and stiffener are as follows:

$$E_1 = 120\,\text{GPa}, \quad E_2 = 7{\cdot}9\,\text{GPa}, \quad G_{12} = 5{\cdot}5\,\text{GPa}, \quad v_{12} = 0{\cdot}33, \quad \rho = 1580\,\text{kg/m}^3.$$

The plate and stiffener dimensions are

$$\text{Plate}: \quad a = 0{\cdot}2\,\text{m}, \quad b = 0{\cdot}2\,\text{m}, \quad t = 0{\cdot}003\,\text{m}.$$

$$\text{Stiffener}: \quad h_s = 0{\cdot}01\,\text{m}, \quad b_s = 0{\cdot}006\,\text{m}.$$

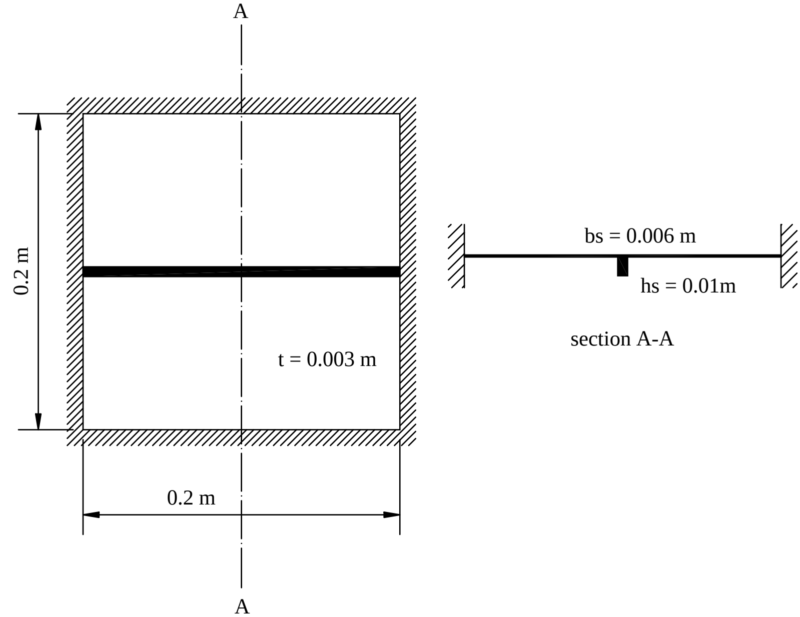

Figure 3. Stiffened laminated plate.

TABLE 4

Natural frequencies of clamped laminated stiffened composite plate with x direction stiffener (Hz)

Mode	Method			
	Super-element [0°]	ANSYS [0°]	Super element [90°]	ANSYS [90°]
1	1357·3	1324·1	891·4	888·5
2	1661·3	1629·7	1349·3	1329·4
3	2099·0	2017·4	2054·4	2017·4
4	2360·1	2303·0	2060·9	2027·6
5	3518·1	3232·6	2308·6	2240·4
6	3566·5	3263·4	3138·2	3052·3

Note: [0°] and [90°] is beam fiber direction.

Two beam lay-ups [0°] and [90°] are considered. The first six natural frequencies are presented in Table 4 and compared with ANSYS as a conventional finite element method.

As indicated in Table 4, very good results are obtained by the super element in comparison with ANSYS. From Table 4, it is clear that different dynamic characteristics of the stiffened plate can be achieved by changing the plate and beam fiber direction.

3.3. CLAMPED LAMINATED STIFFENED PLATE WITH STIFFENER IN THE *Y* DIRECTION

Changing the stiffener of Example 2 to the y direction, results in new dynamic characteristics as shown in Table 5.

TABLE 5

Natural frequencies of clamped laminated stiffened composite plate with y direction stiffener (Hz)

Mode	Method			
	Super-element [0°]	ANSYS [0°]	Super element [90°]	ANSYS [90°]
1	1631·1	1589·8	894·6	891·7
2	1928·2	1876·5	1624·2	1589·8
3	2105·2	2018·3	1943·7	1926·7
4	2547·5	2462·2	2071·1	2018·3
5	3101·7	2882·4	2902·3	2779·4
6	3347·9	3215·0	3034·1	2882·4

Note: [0°] and [90°] is beam fiber direction.

3.4. (2×2) BAY CLAMPED LAMINATED STIFFENED PLATE

Ten layers make up a clamped square plate, stiffened by two eccentric, orthogonally placed and centrally located rectangular beams (Figure 4). The dimensions and properties of the plate and stiffeners are the same as in the previous example.

Considering the fiber direction of the beam lay-up ([0°] and [90°]) four different combination can be achieved:

Case 1: X direction stiffener lay-up: [0°], Y direction stiffener lay-up: [0°]
Case 2: X direction stiffener lay-up: [90°], Y direction stiffener lay-up: [0°]
Case 3: X direction stiffener lay-up: [0°], Y direction stiffener lay-up: [90°]
Case 4: X direction stiffener lay-up: [90°], Y direction stiffener lay-up: [90°].

Results of the analysis for the first six natural mode are presented in Tables 6–9.

The findings in this study, indicate that the super-element is a powerful tool to achieve the same results as the conventional finite element method. Comparing the runtime using

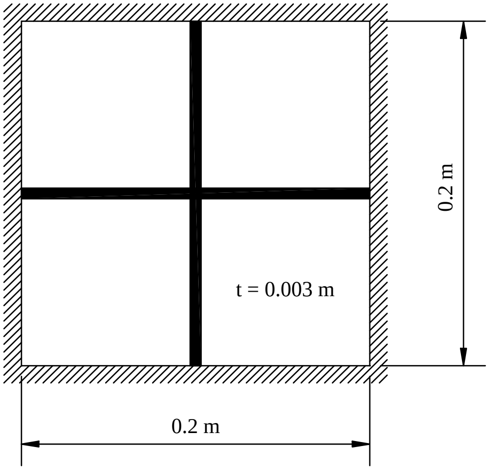

Figure 4. Stiffened laminated plate.

TABLE 6

Natural frequency of clamped stiffened laminated plate of orthogonal stiffener (Hz). Case 1

Mode	Method	
	Super-element	ANSYS
1	2171·4	2045·8
2	2449·0	2323·5
3	2569·7	2496·2
4	2616·5	2496·2
5	3925·5	3704·7

TABLE 7

Natural frequency of clamped stiffened laminated plate of orthogonal stiffener (Hz). Case 2

Mode	Method	
	Super-element	ANSYS
1	2075·4	2029·6
2	2087·0	2045·8
3	2142·1	2056·3
4	2574·0	2478·0
5	3239·7	3082·3
6	3360·8	3228·7

TABLE 8

Natural frequency of clamped stiffened laminated plate of orthogonal stiffener (Hz). Case 3

Mode	Method	
	Super-element	ANSYS
1	1956·5	1927·1
2	2021·5	1996·1
3	2137·8	2045·8
4	2394·8	2323·5
5	3404·9	3262·3
6	3544·4	3369·5

super elements with that of ANSYS shows saving of more than $\frac{2}{3}$ of the conventional runtime.

3.5. (2×2) BAY SIMPLY SUPPORTED LAMINATED STIFFENED PLATE

In the previous case 1 example the supports of the plate are changed to simply supported. This changes the dynamic characteristics of the plates. The obtained results are compared in Table 10.

TABLE 9

Natural frequency of clamped stiffened laminated plate of orthogonal stiffener (Hz). Case 4

Mode	Method	
	Super-element	ANSYS
1	996·4	997·0
2	1946·0	1927·1
3	2059·3	2029·6
4	2108·6	2045·8
5	3009·3	2913·4
6	3184·1	3082·3

TABLE 10

Natural frequency of simply supported stiffened laminated plate of orthogonal stiffener (Hz). Case 1

Mode	Method	
	Super-element	ANSYS
1	1251·3	1245·1
2	1450·3	1399·3
3	1670·4	1618·0
4	1792·4	1734·5
5	2412·7	2351·3

4. CONCLUSION

This paper presents the implementation of a super-finite element method for eccentrically stiffened laminated plates in free vibration. Based on the results obtained from the extensive numerical evaluation, it can be seen that the super-element method has an acceptable level of accuracy for stiffened laminated composite plates with a coarse mesh.

As there are no published results available on the free vibration analysis of laminated stiffened plates, the isotropic plate case has been solved to compare the obtaining results with those previous published findings.

The proposed modelling appears to be well suited for free vibration analysis of laminated stiffened plates. In this regard, owing to its coarse-grid modelling capability, the super element method offers a relatively simple and efficient means of predicting the natural frequencies with much shorter runtime.

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